International Journal of Control, Automation and Systems 16(6) (2018) 2718-2726
http://dx.doi.org/10.1007/s12555-017-0696-8
ISSN:1598-6446 eISSN:2005-4092
http://www.springer.com/12555

Distributed Containment Control of Multi-agent Systems with General Linear Dynamics and Time-delays

Bo Li*, Hong-Yong Yang, Zeng-Qiang Chen, and Zhong-Xin Liu

Abstract: Containment control problems for high-order linear time-invariant multi-agent systems with fixed communication time-delays are investigated. Based on Lyapunov-Krasovskii functional method and the linear matrix inequality (LMI) method, sufcient conditions on the communication digraph, the feedback gains, and the allowed upper bound of the delays to ensure containment control of the multi-agent systems under the different containment control algorithms are given. Finally, numerical simulations are presented to demonstrate theoretical results.

Keywords: Communication time-delays, containment control, general linear dynamics, multi-agent systems.

1. INTRODUCTION

Enlightened by cooperative behaviors in nature, the cooperative control problems have attracted more and more attentions. Recently, more and more researchers are interested in distributed coordination control of multi-agent systems, due to its wide range of applications, such as formation flying of UAVs, and autonomous underwater vehicles. An important feature in distributed cooperative control of multiple agents is that each agent updates its own state based on the information from itself and its local neighbors. Therefore, the consensus is one of the key fundamental problem. The study of consensus algorithms can be dated back to Reynolds [1], Vicsek [2] and Jadbabaie [3]. Detailed information about the recent study of consensus algorithms in cooperative control can be found in Moreau [4] and Ren, Beard [5]. From the perspective of presence and absence of leader(s), the existing investigation about the multi-agent consensus topic can be classied into two subareas. One is the leaderless consensus where leaders are not existent [4–9], and the other subarea is the leader-following consensus [10–12], in which either a single leader or several leaders could be existent. Generally, in the case of multiple leaders, the corresponding problem is called the containment problem, referring to the scenario that the followers will all go asymptotically into the convex hull spanned by the(stationary or dynamic)leaders according to properly designed protocol/algorithm. The initial study of containment can be found in Ji [13]. Many papers have studied the containment control during the next years. In [14], necessary and sufficient conditions for containment control of networked multi-agent systems were given. Cao *et al*. [15] gave the necessary and sufficient conditions for the containment control of multi-agent systems with both stationary and dynamic leaders under fixed and switching directed topology. Yang *et al*. [16] investigated the containment control of heterogeneous fractional-order multi-agent systems. The output containment control problem of multi-agent systems with general linear dynamics was investigated in [17].

In practical multi-agent systems, the delays exist because some reasons. Since the delays may degrade a systems performance or even destroy a systems stability, the investigations have been extensively conducted in this direction [18–21]. When the delays are constant, the frequency-domain approach has been used to give the consensus conditions [20, 21]. The single leader case and/or the leaderless case have been considered in these works. In the study of the containment control problem, some works had been done for the multi-agent systems with delays. The formation containment and containment control problem of the second-order multi-agent systems with time-varying delay was investigated in [22, 23], respectively.

Motivated by the above analysis and note that in some applications, the dynamics of the agents are complicated, and can not be modeled by single or double integrator dynamics. So after our preliminary work [24–26] about containment control of continuous-time, discrete-time and nonlinear multi-agent system with fixed communication

Manuscript received November 8, 2017; revised April 5, 2018; accepted July 22, 2018. Recommended by Associate Editor Kyoungchul Kong under the direction of Editor Yoshito Ohta. This work is supported by the National Natural Science Foundation of China (Nos. 61673200, 61573199, 61573200), Doctoral Scientific Research Foundation of Ludong university (Nos. LB2017029).

Bo Li and Hong-Yong Yang are with the School of Information and Electrical Engineering, Ludong University, Yantai 264025, China (e-mails: libo31680@163.com, hyyang@yeah.net). Zeng-Qiang Chen and Zhong-Xin Liu are with the College of Computer and Control Engineering, Nankai University, Tianjin 300071, China (e-mails: chenzq@nankai.edu.cn, lzhx@nankai.edu.cn).
* Corresponding author.

time-delays, in this paper, we consider the containment control problem for general linear multi-agent systems with time-delays. Using two containment control algorithms, the case with multiple dynamic leaders is investigated. Sufficient conditions on the communication digraph, the feedback gains, and the time-delays to ensure containment control are given.

Compared with the existing works on containment, the novel features of the current paper are twofold. Firstly, the LyapunovKrasovskii functional method and the linear matrix inequality (LMI) method are jointly used. In [27], distributed containment problems were studied, where there do not consider communication time-delays. Secondly, the maximal allowed in LMI can be obtained from the optimization problem that can be solved by using the GEVP solver in Matlabs LMI Toolbox. The paper is organized as follows. In Section 2, we give some basic concepts in graph theory and some relative lemmas. In Section 3, the model is described and the containment control problem of multi-agent system with fixed time-delays is investigated for general linear systems respectively. Numerical simulations and conclusion are given in Sections 4 and 5, respectively.

The following notations are used throughout this paper. The notation $\mathrm{diag}(\omega_1, \omega_2, ..., \omega_n)$ denotes a diagonal matrix whose diagonal entries are and the notation $\mathbf{A}^{\mathrm{T}}$ denotes the transpose of matrix $\mathbf{A}$. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations. The notation $\mathbf{P} > 0$ (≥ 0) means that $\mathbf{P}$ is a real symmetric positive (semi-positive) definite matrix. $\mathbf{I}$ and $\mathbf{0}$ represent, respectively, the identity matrix and zero matrix. The set of real numbers is denoted by $\mathbb{R}$. The set of real-valued vectors of length m is given by $\mathbb{R}^m$. The set of real-valued $m \times n$ matrices is given by $\mathbb{R}^{m\times n}$, and $\otimes$ denotes Kronecker product.

2. PROBLEM FORMULATIONS AND PRELIMINARIES

Let $G(\boldsymbol{V}, \varepsilon, \mathbf{A})$ be a directed graph of order n, with the set of nodes $\boldsymbol{V} = \{v_1, v_2, ..., v_n\}$, and the set of directed edges $\boldsymbol{\varepsilon} \subseteq V \times V$, and a adjacency matrix $\mathbf{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$ with nonnegative adjacency elements a_{ij}. A directed edge ε_{ij} in G is denoted by the ordered pair of node (v_j, v_i), where v_i is defined as the parent node and v_j is defined as the child node, which means that node v_j can receive information from node v_i.The adjacency elements associated with the edges are positive, that is, $\varepsilon_{ij} \in \varepsilon \Leftrightarrow a_{ij} > 0$. Moreover, we assume $a_{ii} = 0$ for all $v_i \in V$. In this paper, we use $N_i(t)$ to denote the neighbor set of agent i at the time t. Correspondingly, we defined the Laplacian matrix $\mathbf{L} = [l_{ij}] \in \mathbb{R}^{n\times n}$of the directed graph as [28]

$$l_{ij} = \begin{cases} \sum_{j=1}^{n} a_{ij}, & i = j, \\ -a_{ij}, & i \neq j. \end{cases}$$

Consider a group of n agents. Suppose that each agent has the general linear dynamics described by

$$\begin{cases} \dot{x}_i(t) = \mathbf{A}x_i(t) + \mathbf{B}u_i(t), \\ y_i(t) = \mathbf{C}x_i(t), \end{cases} \tag{1}$$

where $x_i(t) \in \mathbb{R}^r$, $y_i(t) \in \mathbb{R}^q$ and $u_i(t) \in \mathbb{R}^p$ are the state, the measured output and control input of agent i at time t, respectively.

For the n-agent system, an agent is called a leader if the agent has no neighbor. An agent is called a follower if the agent has a neighbor. Assume that there are m leaders, where $m < n$, and$n-m$ followers. In this paper, we use $\mathfrak{R}$ and $\mathcal{F}$ to denote, respectively, the leader set and the follower set. Without loss of generality, we assume that agents 1 to $n-m$ ($m < n$) are followers and agents $n-m+1$ to n are leaders. Accordingly, L can be partitioned as

$$\mathbf{L} = \begin{bmatrix} \mathbf{L}_1 & \mathbf{L}_2 \\ \mathbf{0}_{m\times(n-m)} & \mathbf{0}_{m\times m} \end{bmatrix},$$

where $\mathbf{L}_1 \in \mathbb{R}^{(n-m)\times(n-m)}$ and $\mathbf{L}_2 \in \mathbb{R}^{(n-m)\times m}$.

Definition 1 [29]**:** Let Q be a set in a real vector space $W \subseteq \mathbb{R}^n$. The set Q is called convex if, for any $\tilde{x}$ and $\tilde{y}$ in Q, the point $(1-\theta)\tilde{x} + \theta\tilde{y}$ is in Q for any $\theta \in [0,1]$. The convex hull for a set of points $\tilde{\mathbf{X}} = \{\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_n\}$ in W is the minimal convex set containing all points in $\tilde{\mathbf{X}}$. We use $\mathrm{Co}(\tilde{\mathbf{X}})$ to denote the convex hull of $\tilde{X}$. In particular,

$$\mathrm{Co}(\tilde{\mathbf{X}}) = \left\{ \sum_{i=1}^{n} a_i\tilde{x}_i | \tilde{x}_i \in \tilde{\mathbf{X}}, a_i \in \mathbf{R}^n \geq 0, \ \sum_{i=1}^{n} a_i = 1 \right\}.$$

Definition 2 [30]**:** The containment control is achieved for the system (1) under a certain control input if the states of the followers asymptotically converge to the convex hull formed by those of the leaders.

Assumption 1 [15]**:** For each follower, there exists at least one leader that has a directed path to the follower.

Lemma 1 [15]**:** Assume the communication digraph G has a directed spanning forest(same as Assumption 1 is satisfied). Then, all the eigenvalues of $\mathbf{L}_1$ have positive real parts, each element of $-\mathbf{L}_1^{-1}\mathbf{L}_2$is nonnegative, and the sum of each row of $-\mathbf{L}_1^{-1}\mathbf{L}_2$ is 1.

From Lemma 1, we know that $\mathbf{L}_1$ is invertible and all eigenvalues of $\mathbf{L}_1$ have positive real parts and not equal to 0.

Lemma 2 [31]**:** Given a positive definite matrix $\mathbf{M} \in \mathbb{R}^{n\times n}$, two constants γ_1 and γ_2 satisfying $\gamma_1 < \gamma_2$, and a vector function $\omega : [\gamma_1, \gamma_2] \to \mathbb{R}^n$ such that the integrations concerned are well defined, the following inequality holds

$$\left(\int_{\gamma_1}^{\gamma_2} \omega(s)ds\right)^{\mathrm{T}} \mathbf{M} \left(\int_{\gamma_1}^{\gamma_2} \omega(s)ds\right) \leq (\gamma_2 - \gamma_1)\left(\int_{\gamma_1}^{\gamma_2} \omega^{\mathrm{T}}(s)\mathbf{M}\omega(s)ds\right).$$

3. MAIN RESULTS

3.1. Containment control of general linear multi-agent systems with communication time-delays

Here we consider the containment control protocol as follows:

$$u_i(t) = \begin{cases} 0, & i \in \Re, \\ -\mathbf{K} \sum_{j \in N_i(t)} a_{ij}[(x_j(t-\tau) - x_i(t-\tau)], & i \in \mathcal{F}. \end{cases} \tag{2}$$

Then the closed-loop system using (2) for (1) can be written as follows:

$$\begin{cases} \dot{\boldsymbol{X}}_{\mathrm{F}}(t) = (\mathbf{I}_{n-m} \otimes \mathbf{A})\boldsymbol{X}_{\mathrm{F}}(t) + (\mathbf{L} \otimes \mathbf{BK})\boldsymbol{X}(t-\tau), \\ \dot{\boldsymbol{X}}_L(t) = (\mathbf{I}_m \otimes \mathbf{A})\boldsymbol{X}_L(t), \end{cases} \tag{3}$$

where $\boldsymbol{X}_{\mathrm{F}}(k) = [x_1(k) \;\; x_2(k) \;\; \ldots \;\; x_{n-m}(k)]^{\mathrm{T}}$, $\boldsymbol{X}_L(k) = [x_{n-m+1}(k) \; \ldots \; x_n(k)]^{\mathrm{T}}$.

We define containment error function as

$$\boldsymbol{E}(t) = \mathbf{X}_{\mathrm{F}}(t) + (\mathbf{L}_1^{-1}\mathbf{L}_2 \otimes \mathbf{I}_m)\boldsymbol{X}_{\mathrm{L}}(t).$$

Then we have

$$\begin{aligned} \dot{\boldsymbol{E}}(t) =& \dot{\boldsymbol{X}}_{\mathrm{F}}(t) + (\mathbf{L}_1^{-1}\mathbf{L}_2 \otimes \mathbf{I}_m)\dot{\boldsymbol{X}}_{\mathrm{L}}(t) \\ =& (\mathbf{I}_{n-m} \otimes \mathbf{A})\boldsymbol{X}_{\mathrm{F}}(t) + (\mathbf{L}_1 \otimes \mathbf{BK})\boldsymbol{X}_{\mathrm{F}}(t-\tau) \\ &+ (\mathbf{L}_2 \otimes \mathbf{BK})\boldsymbol{X}_{\mathrm{L}}(t-\tau) \\ &+ (\mathbf{L}_1^{-1}\mathbf{L}_2 \otimes \mathbf{I}_m)(\mathbf{I}_{n-m} \otimes \mathbf{A})\boldsymbol{X}_{\mathrm{L}}(t) \\ =& (\mathbf{I}_{n-m} \otimes \mathbf{A})\boldsymbol{X}_{\mathrm{F}}(t) + (\mathbf{L}_1 \otimes \mathbf{BK})\boldsymbol{X}_{\mathrm{F}}(t-\tau) \\ &+ (\mathbf{L}_2 \otimes \mathbf{BK})\boldsymbol{X}_{\mathrm{L}}(t-\tau) + (\mathbf{L}_1^{-1}\mathbf{L}_2 \otimes \mathbf{A})\boldsymbol{X}_{\mathrm{L}}(t) \\ =& (\mathbf{I}_{n-m} \otimes \mathbf{A})[\boldsymbol{X}_{\mathrm{F}}(t) + (\mathbf{L}_1^{-1}\mathbf{L}_2 \otimes \mathbf{I}_m)\boldsymbol{X}_{\mathrm{L}}(t)] \\ &+ (\mathbf{L}_1 \otimes \mathbf{BK})[\boldsymbol{X}_{\mathrm{F}}(t-\tau) \\ &+ (\mathbf{L}_1^{-1}\mathbf{L}_2 \otimes \mathbf{I}_m)\boldsymbol{X}_{\mathrm{L}}(t-\tau)] \\ =& (\mathbf{I}_{n-m} \otimes \mathbf{A})\boldsymbol{E}(t) + (\mathbf{L}_1 \otimes \mathbf{BK})\boldsymbol{E}(t-\tau). \end{aligned}$$

The above equation can be rewritten as follows:

$$\dot{\boldsymbol{E}}(t) = \mathbf{G}\boldsymbol{E}(t) + \mathbf{H}\boldsymbol{E}(t-\tau),$$

where $\mathbf{G} = (\mathbf{I}_{n-m} \otimes \mathbf{A})$ and $\mathbf{H} = (\mathbf{L}_1 \otimes \mathbf{BK})$.

Lemma 3: Suppose that (A, B) is stabilizable, (A, C) is detectable, and Assumption 1 holds. Then the matrix all $\mathbf{G}+\mathbf{H}$ is Hurwitz if and only if the matrix $\mathbf{A}+\lambda_k\mathbf{BK}$ is Hurwitz, where λ_k ($k = 1, 2, \ldots, n-m$) is the eigenvalues of $\mathbf{L}_1$.

Proof: It is clear that

$$\mathbf{G}+\mathbf{H} = \mathbf{I}_{n-m} \otimes \mathbf{A} + \mathbf{L}_1 \otimes \mathbf{BK}.$$

Because that the matrix $\mathbf{L}_1$ is invertible, so there is a positive matrix $\mathbf{T}$ satisfying $\mathbf{T}\mathbf{L}_1\mathbf{T}^{-1} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{n-m})$.

Multiplying both sides of the above equation with $(\mathbf{T} \otimes \mathbf{I}_{n-m})$ and $(\mathbf{T} \otimes \mathbf{I}_{n-m})^{-1}$ leads to

$$\begin{aligned} &(\mathbf{T} \otimes \mathbf{I}_{n-m})(\mathbf{G}+\mathbf{H})(\mathbf{T} \otimes \mathbf{I}_{n-m})^{-1} \\ &= (\mathbf{T} \otimes \mathbf{I}_{n-m})(\mathbf{I}_{n-m} \otimes \mathbf{A} + \mathbf{L}_1 \otimes \mathbf{BK})(\mathbf{T} \otimes \mathbf{I}_{n-m})^{-1} \\ &= \mathbf{I}_{n-m} \otimes \mathbf{A} + \mathbf{T}\mathbf{L}_1\mathbf{T}^{-1} \otimes \mathbf{BK} \\ &= \mathbf{I}_{n-m} \otimes \mathbf{A} + \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{n-m}) \otimes \mathbf{BK}. \end{aligned}$$

The proof is completed. □

Suppose that (A, B) is stabilizable, (A, C) is detectable, and Assumption 1 holds. How to get K leads to the matrix $\mathbf{A}+\lambda_k\mathbf{BK}$ is Hurwitz that had been investigated in many papers. In this paper, we use the algorithm as follows:

$$\mathbf{K} = -\max\left\{1, \frac{1}{\min(\mathrm{Re}(\lambda_k))}\right\}\mathbf{B}^{\mathrm{T}}\mathbf{P},$$

where $\mathbf{P}$ is the positive solution of the following algebraic Riccati equations:

$$\mathbf{PA} + \mathbf{A}^{\mathrm{T}}\mathbf{P} + \mathbf{I}_n - \mathbf{PBB}^{\mathrm{T}}\mathbf{P} = 0.$$

Theorem 1: Suppose that (A, B) is stabilizable, (A, C) is detectable and Assumption 1 holds. Protocol (2) asymptotically solves the containment control problem for system (1) with time-delays, if all matrices $\mathbf{A}+\lambda_k\mathbf{BK}$ are Hurwitz and there exists positive definite matrix $\mathbf{P}$, $\mathbf{R}$ and τ satisfying

$$\mathbf{E}_2 + \tau^2\mathbf{F}_2 < 0, \tag{4}$$

where

$$\mathbf{E}_2 = \begin{bmatrix} \mathbf{P}\boldsymbol{\Phi}^{\mathrm{T}} + \boldsymbol{\Phi}\mathbf{P} & -\mathbf{PH} \\ -\mathbf{H}^{\mathrm{T}}\mathbf{P} & -\mathbf{R} \end{bmatrix},$$

$$\mathbf{F}_2 = \begin{bmatrix} \boldsymbol{\Phi}^{\mathrm{T}} \\ -\mathbf{H}^{\mathrm{T}} \end{bmatrix} \mathbf{R} \begin{bmatrix} \boldsymbol{\Phi} & -\mathbf{H} \end{bmatrix}.$$

Proof: Consider the following Lyapunov function candidate

$$\boldsymbol{V}(t) = \boldsymbol{E}^{\mathrm{T}}(t)\mathbf{P}\boldsymbol{E}(t) + \tau\int_{t-\tau}^{t}(s-t+\tau)\dot{\boldsymbol{E}}^{\mathrm{T}}(s)\mathbf{R}\dot{\boldsymbol{E}}(s)ds, \tag{5}$$

where $\mathbf{P}$, $\mathbf{R}$ are positive definite matrices, τ is a positive fixed time-delay, respectively. The time derivative of this Lyapunov candidate along the trajectory of system (1) is

$$\begin{aligned} \dot{\boldsymbol{V}}(t) =& 2\boldsymbol{E}^{\mathrm{T}}(t)\mathbf{P}\dot{\boldsymbol{E}}(t) - \tau\int_{t-\tau}^{t}\dot{\boldsymbol{E}}^{\mathrm{T}}(s)\mathbf{R}\dot{\boldsymbol{E}}(s)ds \\ &+ \tau^2\dot{\boldsymbol{E}}^{\mathrm{T}}(t)\mathbf{R}\dot{\boldsymbol{E}}(t) \\ =& 2\boldsymbol{E}^{\mathrm{T}}(t)\mathbf{P}[\mathbf{G}\boldsymbol{E}(t) + \mathbf{H}\boldsymbol{E}(t-\tau)] \\ &- \tau\int_{t-\tau}^{t}\dot{\boldsymbol{E}}^{\mathrm{T}}(s)\mathbf{R}\dot{\boldsymbol{E}}(s)ds \\ &+ \tau^2[\mathbf{G}\boldsymbol{E}(t) + \mathbf{H}\boldsymbol{E}(t-\tau)]^{\mathrm{T}} \end{aligned}$$

$$\times \mathbf{R}[\mathbf{G}\boldsymbol{E}(t)+\mathbf{H}\boldsymbol{E}(t-\tau)]. \tag{6}$$

We define $\bar{\boldsymbol{E}}(t)=\boldsymbol{E}(t)-\boldsymbol{E}(t-\tau)$ and $\boldsymbol{\Phi}=\mathbf{G}+\mathbf{H}$, then (6) can be rewritten as

$$\begin{aligned}\dot{\boldsymbol{V}}(t)=&2\boldsymbol{E}^{\mathrm{T}}(t)\mathbf{P}[\boldsymbol{\Phi}\boldsymbol{E}(t)-\mathbf{H}\bar{\boldsymbol{E}}(t)]\\&-\tau\int_{t-\tau}^{t}\dot{\boldsymbol{E}}^{\mathrm{T}}(s)\mathbf{R}\dot{\boldsymbol{E}}(s)ds\\&+\tau^2[\boldsymbol{\Phi}\boldsymbol{E}(t)-\mathbf{H}\bar{\boldsymbol{E}}(t)]^{\mathrm{T}}\mathbf{R}[\boldsymbol{\Phi}\boldsymbol{E}(t)-\mathbf{H}\bar{\boldsymbol{E}}(t)]\\=&2\boldsymbol{E}^{\mathrm{T}}(t)\mathbf{P}\boldsymbol{\Phi}\boldsymbol{E}(t)-2\boldsymbol{E}^{\mathrm{T}}(t)\mathbf{P}\mathbf{H}\bar{\boldsymbol{E}}(t)\\&-\tau\int_{t-\tau}^{t}\dot{\boldsymbol{E}}^{\mathrm{T}}(s)\mathbf{R}\dot{\boldsymbol{E}}(s)ds\\&+\tau^2[\boldsymbol{\Phi}\boldsymbol{E}(t)-\mathbf{H}\bar{\boldsymbol{E}}(t)]^{\mathrm{T}}\mathbf{R}[\boldsymbol{\Phi}\boldsymbol{E}(t)-\mathbf{H}\bar{\boldsymbol{E}}(t)].\end{aligned} \tag{7}$$

By Lemma 2,

$$\bar{\boldsymbol{E}}^{\mathrm{T}}(t)\mathbf{R}\bar{\boldsymbol{E}}(t)\le\tau\int_{t-\tau}^{t}\dot{\boldsymbol{E}}^{\mathrm{T}}(s)\mathbf{R}\dot{\boldsymbol{E}}(s)ds.$$

It follows that

$$\begin{aligned}\dot{\boldsymbol{V}}(t)\le&\begin{bmatrix}\boldsymbol{E}(t)\\\bar{\boldsymbol{E}}(t)\end{bmatrix}^{\mathrm{T}}\left(\begin{bmatrix}\mathbf{P}\boldsymbol{\Phi}^{\mathrm{T}}+\boldsymbol{\Phi}\mathbf{P}&-\mathbf{P}\mathbf{H}\\-\mathbf{H}^{\mathrm{T}}\mathbf{P}&-\mathbf{R}\end{bmatrix}\right.\\&\left.+\tau^2\begin{bmatrix}\boldsymbol{\Phi}^{\mathrm{T}}\\-\mathbf{H}^{\mathrm{T}}\end{bmatrix}\mathbf{R}\begin{bmatrix}\boldsymbol{\Phi}&-\mathbf{H}\end{bmatrix}\right)\begin{bmatrix}\boldsymbol{E}(t)\\\bar{\boldsymbol{E}}(t)\end{bmatrix}.\end{aligned} \tag{8}$$

In view of (4), we know $\dot{\boldsymbol{V}}(t)<0$, which implies that $\lim_{t\to\infty}\boldsymbol{E}(t)=0$, that is, the containment control of system (1) under protocol (2) is achieved. The proof is completed. □

Remark 1: It is clear that $\mathbf{F}_2\ge0$ and $\gamma\tau^2>0$. So (11) maybe hold only when the $\mathbf{E}_2<0$ holds. And (4) can hold when τ is sufficiently close to zero if $\mathbf{E}_2<0$ holds.

Remark 2: By the Schur complement theorem, the condition that $\mathbf{E}_2<0$ holds if and only if $\boldsymbol{\Omega}<0$ holds. The condition that $\boldsymbol{\Omega}<0$ holds only when $\mathbf{P}\boldsymbol{\Phi}^{\mathrm{T}}+\boldsymbol{\Phi}\mathbf{P}<0$ holds. So it is worth pointing out that a necessary condition for (4) is that Assumption 1 holds. If Assumption 1 does not hold, $\mathbf{L}_1$ must have at least one zero eigenvalue and yields a contradiction.

Remark 3: In this paper, all agents are assumed to be the same, output containment control of linear heterogeneous multi-agent systems have been investigated in [32]. When the communication time-delay between agents is varying and the condition $0<\tau(t)\le\tau_d$ holds, the fixed time-delay τ will be replaced with the upper bound of the varying time-delay τ_d in (4). Furthermore, heterogeneous delay problems are difficult to be handled in multi-agent systems, containment control problems for general linear multi-agent systems with heterogeneous delays are still open.

3.2. Containment control of general linear multi-agent systems with observer-based protocols

In this section, a cooperative dynamic regulator is proposed that uses only the neighbors' output measurements of each agent. Here we propose the following containment control protocol:

$$u_i(t)=\begin{cases}0, & i\in\Re,\\ \mathbf{K}\sum_{j\in N_i(t)}a_{ij}[(\hat{x}_i(t-\tau)-\hat{x}_j(t-\tau)], & i\in\mathcal{F},\end{cases} \tag{9}$$

where $\hat{x}_i(t)$ is the estimate of the state $x_i(t)$. And it is defined as follows:

$$\begin{aligned}\dot{\hat{x}}_i(t)=&\mathbf{A}\hat{x}_i(t)+\mathbf{B}u_i(t)\\&+\mathbf{F}\mathbf{C}\sum_{j\in N_i(t)}a_{ij}((\hat{x}_i(t-\tau)-\hat{x}_j(t-\tau))\\&-\mathbf{F}\sum_{j\in N_i(t)}a_{ij}((y_i(t-\tau)-y_j(t-\tau)).\end{aligned} \tag{10}$$

Under protocol (9), the multi-agent system (1) can be written in a compact form as follows:

$$\dot{x}_i(t)=\begin{cases}\mathbf{A}x_i(t), & i\in\Re\\ \mathbf{A}x_i(t)\\ +\mathbf{B}\mathbf{K}\sum_{j\in N_i(t)}a_{ij}[(\hat{x}_i(t-\tau)-\hat{x}_j(t-\tau)], & i\in\mathcal{F}.\end{cases} \tag{11}$$

We define the state estimation error for agent i

$$e_i=\hat{x}_i(t)-x_i(t). \tag{12}$$

Then for the followers, we have

$$\begin{aligned}\dot{e}_i(t)=&\dot{\hat{x}}_i(t)-\dot{x}_i(t)\\=&\mathbf{A}\hat{x}_i(t)+\mathbf{B}u_i(t)\\&+\mathbf{F}\mathbf{C}\sum_{j\in N_i(t)}a_{ij}((\hat{x}_i(t-\tau)-\hat{x}_j(t-\tau))\\&-\mathbf{F}\sum_{j\in N_i(t)}a_{ij}\Big((y_i(t-\tau)-y_j(t-\tau))\\&-\mathbf{A}x_i(t)-\mathbf{B}u_i(t)\Big)\\=&\mathbf{A}e_i(t)+\mathbf{F}\mathbf{C}\sum_{j\in N_i(t)}a_{ij}((e_i(t-\tau)-e_j(t-\tau)),\\&i\in F,\end{aligned} \tag{13}$$

and for the leaders, we have

$$\dot{\hat{x}}_i(t)=\mathbf{A}\hat{x}_i(t)\ \Rightarrow\ e_i(t)=0,\ \ i\in\Re \tag{14}$$

We define

$$\begin{cases}e_{\mathrm{L}}(t)=[e_1(t)^{\mathrm{T}},e_2(t)^{\mathrm{T}},\dots,e_m(t)^{\mathrm{T}}]^{\mathrm{T}},\\ e_{\mathrm{F}}(t)=[e_{m+1}(t)^{\mathrm{T}},e_{m+2}(t)^{\mathrm{T}},\dots,e_n(t)^{\mathrm{T}}]^{\mathrm{T}}.\end{cases}$$

Further, we have

$$\begin{cases}e_{\mathrm{L}}(t)=0,\\ \dot{e}_{\mathrm{F}}(t)=(\mathbf{I}_{n-m}\otimes\mathbf{A})e_{\mathrm{F}}(t)+(\mathbf{L}\otimes\mathbf{F}\mathbf{C})e(t-\tau)\\ \qquad=(\mathbf{I}_{n-m}\otimes\mathbf{A})e_{\mathrm{F}}(t)+(\mathbf{L}_1\otimes\mathbf{F}\mathbf{C})e_{\mathrm{F}}(t-\tau)\\ \qquad\quad+(\mathbf{L}_2\otimes\mathbf{F}\mathbf{C})e_{\mathrm{L}}(t-\tau)\\ \qquad=(\mathbf{I}_{n-m}\otimes\mathbf{A})e_{\mathrm{F}}(t)+(\mathbf{L}_1\otimes\mathbf{F}\mathbf{C})e_{\mathrm{F}}(t-\tau).\end{cases} \tag{15}$$

Then (11) can be rewritten

$$\begin{aligned}\dot{x}_i(t) =&\mathbf{A}x_i(t)+\mathbf{BK}\sum_{j\in N_i(t)} a_{ij}[(e_i(t-\tau)-e_j(t-\tau)]\\&+\mathbf{BK}\sum_{j\in N_i(t)} a_{ij}[(x_i(t-\tau)-x_j(t-\tau)]\\=&\mathbf{A}x_i(t)+\mathbf{BKL}e(t-\tau)+\mathbf{BKLX}(t-\tau).\end{aligned}\quad (16)$$

From (1) and protocol (9), the closed-loop networked dynamics can be written as

$$\begin{aligned}\dot{\boldsymbol{X}}_{\mathrm{F}}(t) =&(\mathbf{I}_{n-m}\otimes\mathbf{A})\boldsymbol{X}_{\mathrm{F}}(t)+(\mathbf{L}_1\otimes\mathbf{BK})\boldsymbol{X}_{\mathrm{F}}(t-\tau)\\&+(\mathbf{L}_2\otimes\mathbf{BK})\boldsymbol{X}_{\mathrm{L}}(t-\tau)+(\mathbf{L}_1\otimes\mathbf{BK})e_{\mathrm{F}}(t-\tau)\\&+(\mathbf{L}_2\otimes\mathbf{BK})e_{\mathrm{L}}(t-\tau)\\=&(\mathbf{I}_{n-m}\otimes\mathbf{A})\boldsymbol{X}_{\mathrm{F}}(t)+(\mathbf{L}_1\otimes\mathbf{BK})\boldsymbol{X}_{\mathrm{F}}(t-\tau)\\&+(\mathbf{L}_2\otimes\mathbf{BK})\boldsymbol{X}_{\mathrm{L}}(t-\tau)+(\mathbf{L}_1\otimes\mathbf{BK})e_{\mathrm{F}}(t-\tau).\end{aligned}\quad (17)$$

Here we use the same definition of containment error function

$$\boldsymbol{E}(t)=\boldsymbol{X}_{\mathrm{F}}(t)+(\mathbf{L}_1^{-1}\mathbf{L}_2\otimes\mathbf{I}_m)\boldsymbol{X}_{\mathrm{L}}(t).$$

Then, we have

$$\begin{aligned}\dot{\boldsymbol{E}}(t) =&\dot{\boldsymbol{X}}_{\mathrm{F}}(t)+(\mathbf{L}_1^{-1}\mathbf{L}_2\otimes\mathbf{I}_m)\dot{\boldsymbol{X}}_{\mathrm{L}}(t)\\=&(\mathbf{I}_{n-m}\otimes\mathbf{A})\boldsymbol{X}_{\mathrm{F}}(t)+(\mathbf{L}_1\otimes\mathbf{BK})\boldsymbol{X}_{\mathrm{F}}(t-\tau)\\&+(\mathbf{L}_2\otimes\mathbf{BK})\boldsymbol{X}_{\mathrm{L}}(t-\tau)+(\mathbf{L}_1\otimes\mathbf{BK})e_{\mathrm{F}}(t-\tau)\\&+(\mathbf{L}_1^{-1}\mathbf{L}_2\otimes\mathbf{I}_m)(\mathbf{I}_{n-m}\otimes\mathbf{A})X_{\mathrm{L}}(t)\\=&(\mathbf{I}_{n-m}\otimes\mathbf{A})\boldsymbol{X}_{\mathrm{F}}(t)+(\mathbf{L}_1\otimes\mathbf{BK})\boldsymbol{X}_{\mathrm{F}}(t-\tau)\\&+(\mathbf{L}_2\otimes\mathbf{BK})\boldsymbol{X}_{\mathrm{L}}(t-\tau)+(\mathbf{L}_1^{-1}\mathbf{L}_2\otimes\mathbf{A})\boldsymbol{X}_{\mathrm{L}}(t)\\&+(\mathbf{L}_1\otimes\mathbf{BK})e_{\mathrm{F}}(t-\tau)\\=&(\mathbf{I}_{n-m}\otimes\mathbf{A})[\boldsymbol{X}_{\mathrm{F}}(t)+(\mathbf{L}_1^{-1}\mathbf{L}_2\otimes\mathbf{I}_m)\boldsymbol{X}_{\mathrm{L}}(t)]\\&+(\mathbf{L}_1\otimes\mathbf{BK})[\boldsymbol{X}_{\mathrm{F}}(t-\tau)+(\mathbf{L}_1^{-1}\mathbf{L}_2\otimes\mathbf{I}_m)\boldsymbol{X}_{\mathrm{L}}(t-\tau)]\\&+(\mathbf{L}_1\otimes\mathbf{BK})e_{\mathrm{F}}(t-\tau)\\=&(\mathbf{I}_{n-m}\otimes\mathbf{A})\boldsymbol{E}(t)+(\mathbf{L}_1\otimes\mathbf{BK})\boldsymbol{E}(t-\tau)\\&+(\mathbf{L}_1\otimes\mathbf{BK})e_{\mathrm{F}}(t-\tau).\end{aligned}\quad (18)$$

Equation (18) can be written

$$\begin{aligned}\dot{\boldsymbol{E}}(t) =&(\mathbf{I}_{n-m}\otimes\mathbf{A})\boldsymbol{E}(t)+(\mathbf{L}_1\otimes\mathbf{BK})\boldsymbol{E}(t-\tau)\\&+(\mathbf{L}_1\otimes\mathbf{BK})e_{\mathrm{F}}(t-\tau).\end{aligned}\quad (19)$$

Then, multi-agent system (11) can be rewritten as follows:

$$\begin{aligned}\begin{bmatrix}\dot{\boldsymbol{E}}(t)\\ \dot{e}_{\mathrm{F}}(t)\end{bmatrix} =&\mathbf{I}_{n-m}\otimes\begin{bmatrix}\mathbf{A} & \mathbf{0}\\ \mathbf{0} & \mathbf{A}\end{bmatrix}\begin{bmatrix}\boldsymbol{E}(t)\\ e_{\mathrm{F}}(t)\end{bmatrix}\\&+\begin{bmatrix}\mathbf{L}_1\otimes\mathbf{BK} & \mathbf{L}_1\otimes\mathbf{BK}\\ \mathbf{0} & \mathbf{L}_1\otimes\mathbf{FC}\end{bmatrix}\begin{bmatrix}\boldsymbol{E}(t-\tau)\\ e_{\mathrm{F}}(t-\tau)\end{bmatrix}.\end{aligned}\quad (20)$$

We define

$$\boldsymbol{\chi}(t)=\begin{bmatrix}\boldsymbol{E}(t)\\ e_{\mathrm{F}}(t)\end{bmatrix},\ \Upsilon=\mathbf{I}_{n-m}\otimes\begin{bmatrix}\mathbf{A} & \mathbf{0}\\ \mathbf{0} & \mathbf{A}\end{bmatrix},$$

$$\boldsymbol{\Xi}=\begin{bmatrix}\mathbf{L}_1\otimes\mathbf{BK} & \mathbf{L}_1\otimes\mathbf{BK}\\ \mathbf{0} & \mathbf{L}_1\otimes\mathbf{FC}\end{bmatrix}.$$

Equation (20) can be rewritten

$$\dot{\boldsymbol{\chi}}(t)=\Upsilon\boldsymbol{\chi}(t)+\boldsymbol{\Xi}\boldsymbol{\chi}(t-\tau).\quad (21)$$

Theorem 2: Suppose that (A, B) is stabilizable, (A, C) is detectable and Assumption 1 holds. Protocol (9) asymptotically solves the containment control problem for system (1) with time-delays, if all matrices $\mathbf{A}+\lambda_k\mathbf{BK}$ and $\mathbf{A}+\lambda_k\mathbf{FC}$ are Hurwitz, and there exists positive de?nite matrix $\mathbf{P}$, $\mathbf{R}$ and τ satisfying

$$\tilde{\mathbf{E}}_2+\tau^2\tilde{\mathbf{F}}_2<0,\quad (22)$$

where

$$\tilde{\mathbf{E}}_2=\begin{bmatrix}\mathbf{P}\tilde{\boldsymbol{\Phi}}^{\mathrm{T}}+\tilde{\boldsymbol{\Phi}}\mathbf{P} & -\mathbf{P}\boldsymbol{\Xi}\\ -\boldsymbol{\Xi}^{\mathrm{T}}\mathbf{P} & -\mathbf{R}\end{bmatrix},$$

$$\tilde{\mathbf{F}}_2=\begin{bmatrix}\tilde{\boldsymbol{\Phi}}^{\mathrm{T}}\\ -\boldsymbol{\Xi}^{\mathrm{T}}\end{bmatrix}\mathbf{R}\begin{bmatrix}\tilde{\boldsymbol{\Phi}} & -\boldsymbol{\Xi}\end{bmatrix},$$

$$\tilde{\boldsymbol{\Phi}}=\boldsymbol{\Xi}+\Upsilon.$$

The proof procedure is basically the same as what had been done in Theorem 1. In order to avoid duplication, it is omitted.

Remark 4: We use the following algorithm to get matrix $\mathbf{F}$ that can satisfy $\mathbf{A}+\lambda_k\mathbf{FC}$ is Hurwitz.

$$\mathbf{F}=-\max\left\{1,\frac{1}{\min(\mathrm{Re}(\lambda_k))}\right\}\mathbf{PC}^{\mathrm{T}},$$

where P is the positive solution of the following algebraic Riccati equations:

$$\mathbf{PA}+\mathbf{A}^{\mathrm{T}}\mathbf{P}+\mathbf{I}_n-\mathbf{PC}^{\mathrm{T}}\mathbf{CP}=0.$$

4. NUMERRICAL SIMULATIONS

In this section, we present several simulation results to validate the previous theoretical results. We consider a group of agents with 4 leaders and 4 followers. When the directed graph $G$is fixed as in Fig. 1, it can be noted that Assumption 1 is satisfied.

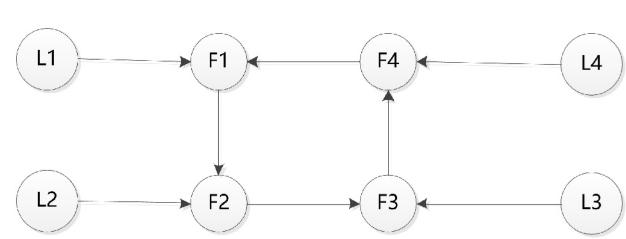

Fig. 1. Fixed directed network topology.

Example 1: Assume that

$$\mathbf{A}=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix},\quad \mathbf{B}=\begin{bmatrix}0\\ 1\end{bmatrix},\quad \mathbf{C}=\begin{bmatrix}1 & 0\end{bmatrix}.$$

By using Matlab Riccati Toolbox, we can get that

$$\mathbf{P}_1=\begin{bmatrix}13.4102 & 5.2745\\ 5.27450 & 4.5425\end{bmatrix},$$
$$\mathbf{K}=\begin{bmatrix}6.4222 & 5.5309\end{bmatrix}.$$

Solving the LMIs(4) by using the GEVP solver in Matlabs LMI Toolbox, we can get τ and $\tau_{op}=0.0335$.

$$\mathbf{P}=1.0e-007$$
$$*\begin{bmatrix}0.4013 & 0.1580 & 0.2511 & 0.1064 & 0.4013\\ 0.1580 & 0.1267 & 0.1061 & 0.0833 & 0.1580\\ 0.2511 & 0.1061 & 0.3801 & 0.1475 & 0.2511\\ 0.1064 & 0.0833 & 0.1475 & 0.1166 & 0.1064\\ 0.4013 & 0.1580 & 0.2511 & 0.1064 & 0.4013\\ 0.1580 & 0.1267 & 0.1061 & 0.0833 & 0.1580\\ 0.2511 & 0.1061 & 0.3801 & 0.1475 & 0.2511\\ 0.1064 & 0.0833 & 0.1475 & 0.1166 & 0.1064\end{bmatrix.$$
$$\begin{matrix}0.1580 & 0.2511 & 0.1064\\ 0.1267 & 0.1061 & 0.0833\\ 0.1061 & 0.3801 & 0.1475\\ 0.0833 & 0.1475 & 0.1166\\ 0.1580 & 0.2511 & 0.1064\\ 0.1267 & 0.1061 & 0.0833\\ 0.1061 & 0.3801 & 0.1475\\ 0.0833 & 0.1475 & 0.1166\end{bmatrix},$$

$$\mathbf{R}=1.0e-006$$
$$*\begin{bmatrix}0.2407 & -0.0230 & 0.1030 & -0.0404\\ -0.0230 & 0.2858 & -0.0253 & 0.1350\\ 0.1030 & -0.0253 & 0.2422 & -0.0303\\ -0.0404 & 0.1350 & -0.0303 & 0.2805\\ 0.2407 & -0.0230 & 0.1030 & -0.0404\\ -0.0230 & 0.2858 & -0.0253 & 0.1350\\ 0.1030 & -0.0253 & 0.2422 & -0.0303\\ -0.0404 & 0.1350 & -0.0303 & 0.2805\end{matrix}$$
$$\begin{matrix}0.2407 & -0.0230 & 0.1030 & -0.0404\\ -0.0230 & 0.2858 & -0.0253 & 0.1350\\ 0.1030 & -0.0253 & 0.2422 & -0.0303\\ -0.0404 & 0.1350 & -0.0303 & 0.2805\\ 0.2407 & -0.0230 & 0.1030 & -0.0404\\ -0.0230 & 0.2858 & -0.0253 & 0.1350\\ 0.1030 & -0.0253 & 0.2422 & -0.0303\\ -0.0404 & 0.1350 & -0.0303 & 0.2805\end{bmatrix}.$$

The simulation result using (2) for (1) is shown in Figs. 2-4 when $\tau=0.03$, $\tau=0.08$, respectively.

We can see that states x_1, x_2 of all followers ultimately converge to the convex hull formed by the dynamic leaders when the time-delays $\tau=0.03$. However, it fails to converge when the time-delays $\tau=0.08$.

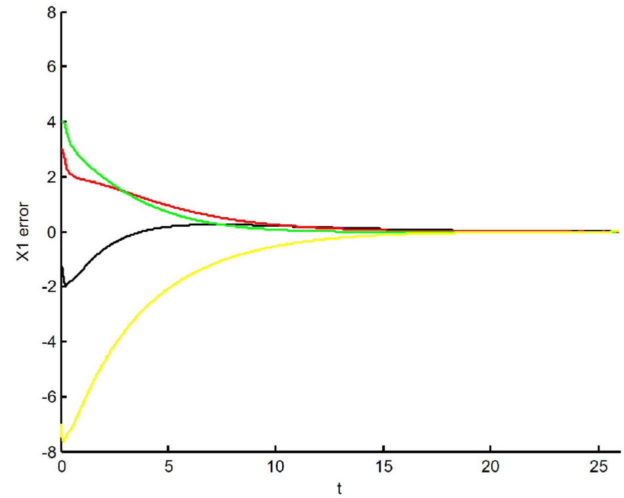

Fig. 2. State x_1 error trajectories of system (1) ($\tau=0.03$).

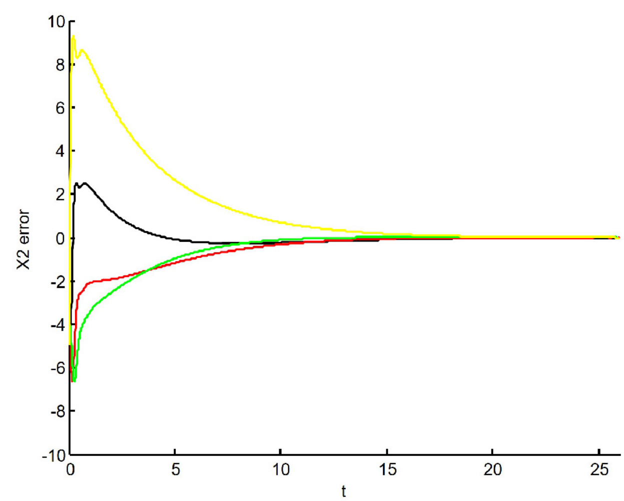

Fig. 3. State x_2 trajectories of system (1) ($\tau=0.03$).

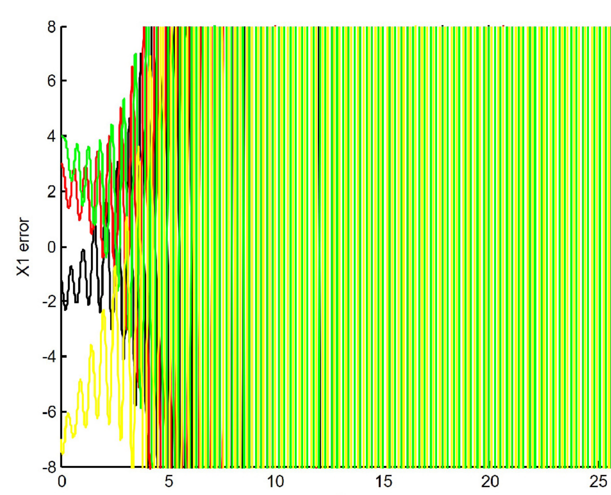

Fig. 4. State x_1 trajectories of system (1) ($\tau=0.08$).

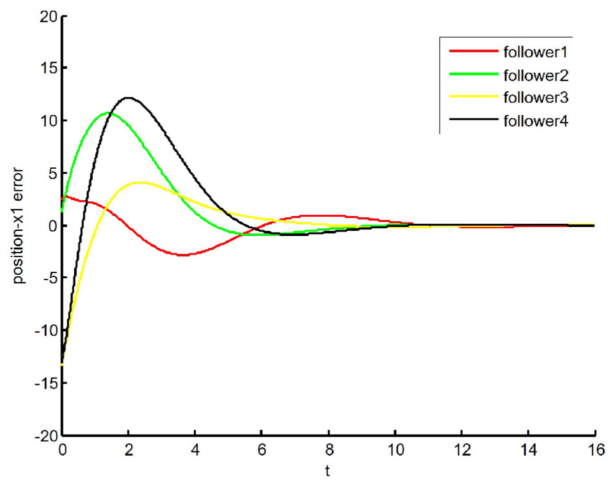

Fig. 5. State x_1 error trajectories of system (1) ($\tau = 0.05$).

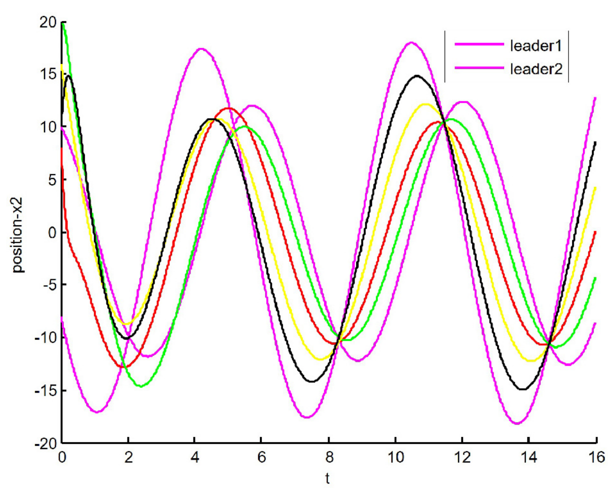

Fig. 6. State x_2 trajectories of system (1) ($\tau = 0.05$).

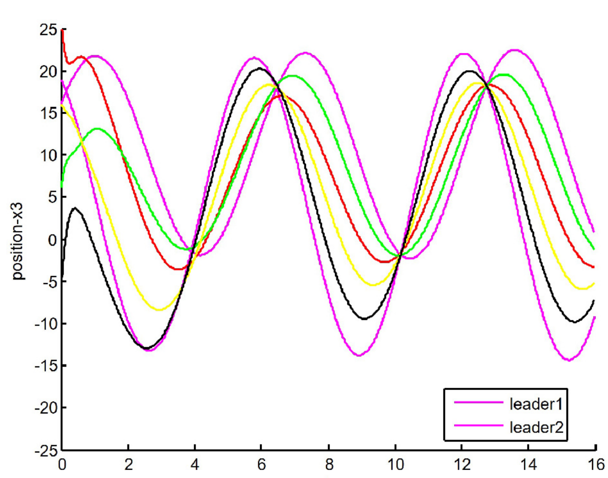

Fig. 7. State x_3 trajectories of system (1) ($\tau = 0.05$).

Example 2: Assume that

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \ \mathbf{C} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

By using Matlab Riccati Toolbox, we can get that

$$\mathbf{P}_1 = 1.0e+007 * \begin{bmatrix} 6.7330 & -0.0000 & -6.7330 \\ -0.0000 & 0.0000 & 0.0000 \\ -6.7330 & 0.0000 & 6.7330 \end{bmatrix}.$$

We can obtain $\mathbf{K} = -[0.2999 \ \ 1.1447 \ \ 1.0000]$ and

$$\mathbf{F} = \begin{bmatrix} -2.6742 & 0.7980 \\ -0.3030 & -3.2751 \\ -1.3371 & -1.1010 \end{bmatrix}.$$

Solving the LMIs(22) by using the GEVP solver in Matlabs LMI Toolbox, we can get

$$alpha = 312.8883 \ \Rightarrow \ \tau_{op} = 0.0565.$$

The simulation result using protocol (7) for system (1) is shown in Fig. 5, Fig. 6, and Fig. 7 respectively. We can see that all states x_1, x_2, x_3 of all followers ultimately converge to the convex hull formed by the dynamic leaders when the time-delay $\tau = 0.05$.

5. CONCLUSION

This paper studied the distributed containment control problem of multi-agent systems with general linear dynamic and fixed communication time-delays under fixed directed network topologies. Two kind of different containment control protocol considering communication time-delays are investigated. We showed sufficient conditions on the directed network topology and time-delay to guarantee distributed containment control. In this paper, consensus-based approaches were used to transform containment control problems into the asymptotic stability problems and a Lyapunov-Krasovskii functional approach was used to analyze the stability problems. This method can also be applied to the formation or formation-containment investigation results for the multi-agent systems with time-delays.

There are still a number of related interesting problems deserving further investigation. Containment control problems for heterogeneous multi-agent systems or multi-agent systems with varying time-delays will be discussed in our future work. Meanwhile, it is desirable to study co-operative containment control of linear multi-agent with sampled data.

REFERENCES

[1] C. W. Reynolds, "Flocks, herds and schools: A distributed behavioral model," *Computer Graphics*, vol. 21, no. 4, pp. 25-34, 1987.

[2] T. Vicsek, A. Czirok, E. B. Jacob, I. Cohen, and O. Schochet, "Novel type of phase transition in a system of self-driven particles," *Physical Review Letters*, vol. 75, no. 6,pp. 1226-1229, 1995.

[3] A. Jadbabaie, J. Lin, and A. S. Morse, "Coordination of groups of mobile agents using nearest neighbor rules," *IEEE Trans. on Automatic Control*, vol. 48, no. 6, pp. 988-1001, 2003.

[4] L. Moreau, "Stability of multi-agent systems with time dependent communication links," *IEEE Trans. on Automatic Control*, vol. 50, no. 2, pp. 169-182, 2005.

[5] W. Ren and R. W. Beard, "Consensus seeking in multi-agent systems under dynamically changing interaction topologies," *IEEE Trans. on Automatic Control*, vol. 50, no. 5, pp. 655-661, 2005.

[6] W. Ren, "On consensus algorithms for double-integrator dynamics," *IEEE Trans. on Automatic Control*, vol. 53, no. 6, pp. 1503-1509, 2008.

[7] S. Li, H. Du, and X. Lin, "Finite-time consensus algorithm for multi-agent systems with double-integrator dynamics,? *Automatica*, vol. 47, no. 8, pp. 1706-1712, Aug. 2011.

[8] J. Qin and H. Gao, "A suf?cient condition for convergence of sampled-data consensus for double-integrator dynamics with non uniform and time-varying communication delays," *IEEE Trans. on Automatic Control*, vol. 57,no. 9, pp. 2417-2422, Sep. 2012.

[9] L. Song, D. Huang, S. K. Nguang, and S. Fu, "Mean Square Consensus of Multi-agent Systems with Multiplicative Noises and Time Delays under Directed Fixed Topologies," *International Journal of Control, Automation, and Systems*, vol. 14, no. 1, pp. 69-77, February 2016.

[10] J. Hu and G. Feng, "Distributed tracking control of leader-follower multi-agent systems under noisy measurement," *Automatica*, vol. 46, no. 8, pp. 1382-1387, 2010.

[11] Z. Lin, W. Ding, G. Yan, C. Yu, and A. Giua, "Leader-follower formation via complex Laplacian," *Automatica*, vol. 49, no. 6, pp. 1900-1906, 2013.

[12] J. Luo and C. Cao, "Consensus in multi-agent systems with nonlinear uncertainties under a fixed undirected graph," *International Journal of Control, Automation, and Systems*, vol. 12, no. 2, pp. 231-240, April 2014.

[13] M. Ji, G. Ferrari-Trecate, M. Egerstedt, and A. Buffa, "Containment control in mobile networks," *IEEE Trans. on Automatic Control*, vol. 53, no. 8, pp. 1972-1975, 2008.

[14] H. Liu, G. M. Xie, and L. Wang, "Necessary and sufficient conditions for containment control of networked multi-agent systems," *Automatica*, vol. 48, no. 7, pp. 1415-1422, 2012.

[15] Y. C. Cao, W. Ren, and M. Egerstedt, "Distributed containment control with multiple stationary or dynamic leaders in fixed and switching directed networks," *Automatica*, vol. 48, no. 8, pp. 1586-1597, 2012.

[16] H. Y. Yang, F. J. Han, and M. Zhao, "Distributed containment control of heterogeneous fractional-order multi-agent systems with communication delays," *Open Physics*, vol. 15, no. 1, pp. 509-516, 2017.

[17] X. W. Dong, F. Meng, Z. Shi, G. Lu, and Y. Zhong, "Output containment control for swarm systems with general linear dynamics: A dynamic output feedback approach," *Systems & Control Letters*, vol. 71, pp. 31-37, 2014.

[18] P. Lin, Y.-M Jia, and J.-P Du, "Distributed control of multi-agent systems with second-order agent dynamics and delay-dependent communications," *Asian Journal of Control*, vol. 10, no. 2, pp. 254-259, 2008.

[19] P. Lin, and Y. M. Jia, "Average consensus in networks of multi-agents with both switching topology and coupling time-delay," *Physica A: Statistical Mechanics and its Applications*, vol. 387, no. 1, pp. 303-313, 2008.

[20] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Trans. on Automatic Control*, vol. 49, no. 9, pp. 1520-1533 , 2004.

[21] W.-W Yu, G.-R Chen, and M. Cao, "Some necessary and sufficient conditions for second-order consensus in multi-agent dynamical systems," *Automatica*, vol. 46, no. 6, pp. 1089-1095, 2010.

[22] K.-E Liu, G.-M Xie, and L. Wang, "Containment control for second-order multi-agent systems with time-varying delays," *Systems & Control Letters*, vol. 67, pp. 24-31, 2014.

[23] L. Han, X. W. Dong, Q. D. Li, and Z. Ren, "Formation-containment control for second-order multi-agent systems with time-varying delays," *Neurocomputing*, vol. 218, pp. 439-447, 2016.

[24] B. Li, Z. Q. Chen, Z. X. Liu, and C. Y. Zhang, "Containment control of discrete-time multi-agent systems with multiple stationary leaders and time-delays," *Proceedings of the 34th Chinese Control Conference*, HangZhou, China, July 28-30, pp. 7062-7066, 2015.

[25] B. Li, Z. Q. Chen, Z. X. Liu, C. Y. Zhang, Q. Zhang, "Containment control of multi-agent systems with fixed time-delays in fixed directed networks," *Neurocomputing*, vol. 173, pp. 2069-2075, 2016.

[26] B. Li, Z. Q. Chen, Z. X. Liu, C. Y. Zhang, and Q. Zhang, "Containment control for directed networks multi-agent system with nonlinear dynamics and communication time-delays," *International Journal of Control, Automation, and Systems*, vol. 15, no. 3, pp. 1181-1188, 2017.

[27] Q. Ma and G. Miao, "Distributed containment control of linear multi-agent systems," *Neurocomputing*, vol. 133, pp. 399-403, 2014.

[28] W. Ren and Y. C. Cao, *Distributed Coordination of Multi-agent Networks*, Springer, London, 2011.

[29] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, New Jersey, 1972.

[30] Z. Li, W. Ren, X. Liu, and M. Fu, "Distributed containment control of multi-agent systems with general linear dynamics in the presence of multiple leaders," *International Journal of Robust and Nonlinear Control*, vol. 23, pp. 534-547, 2013.

[31] K. Gu, VL. Kharitonov, and J. Chen, *Stability of Time-Delay Systems*, Birkhauser, 2003.

[32] S. Zuo, Y. Song, F. L. Lewis, and A. Davoudi, "Output containment control of linear heterogeneous multi-agent systems using internal model principle," *IEEE Trans. on Cybernetics*, vol. 47, no. 8, pp. 2099-2109 , 2017.

Bo Li received the B.S. degree in computer application, M.S. degree in computer system structure from the Shandong University, Jinan, China, in 1994 and 2006, respectively, received his Ph.D. degree in Control Theory and Control Engineering from Nankai University, Tianjin, China. He has been at Ludong University. His main areas of research are in containment control of multi-agents system.

Hong-Yong Yang received his Ph.D. degree in Control Theory and Control Engineering from Southeast University in 2005. He is a professor in School of Information and Electrical Engineering, Ludong University. His research interest covers complex network, multi-agent systems, intelligence control.

Zeng-Qiang Chen received the B.S. degree in mathematics, his M.S. and Ph.D. degrees in control theory and control engineering from the Nankai University, Tianjin, China, in 1987, 1990, and 1997, respectively. He has been at Nankai University, where he is currently a professor in the Department of Automation. His main areas of research are in neural network control, complex networks and multi-agents system.

Zhong-Xin Liu received the B.S. degree in Automation and his Ph.D. degree in control theory and control engineering from the Nankai University, Tianjin, China, in 1997 and 2002, respectively. He has been at Nankai University, where he is currently a professor in the Department of Automation. His main areas of research are in predictive control, complex networks and multi-agents system.